H20T3A1

Sei $\Omega := \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1; x_2 > 0\}$ die obere Hälfte der Einheitskreisscheibe. Berechnen Sie

$$\int_\Omega x_1^2 (x_1^2 + x_2^2)^2 dx$$

mit Hilfe von Polarkoordinaten.

Lösung:

$\Omega = \{(x_1, x_2) = (r\cos\varphi, r\ \sin\varphi) \in \mathbb{R}^2 : 0 < r < 1, 0 < \varphi < \pi\}$ in Polarkoordinaten. Somit gilt $\int_\Omega x_1^2(x_1^2 + x_2^2)^2 dx = \int_0^1 \int_0^\pi (r cos\varphi)^2((r cos\varphi)^2 + (r sin\varphi)^2)^2 r\ dr d\varphi =$ $\int_0^1 \int_0^\pi r^2 (cos\varphi)^2\ r^4 r\ dr d\varphi = \int_0^1 r^7 dr \int_0^\pi (cos\varphi)^2 d\varphi = \left[\frac{1}{8} r^8\right]_0^1 \int_0^\pi (cos\varphi)^2 d\varphi =$ $\frac{1}{8}\int_0^\pi (cos\varphi)^2 d\varphi = \frac{\pi}{16}$, denn mittels partieller Integration erhält man $\int_0^\pi (cos\varphi)^2 d\varphi =$ $\int_0^\pi (cos\varphi)(cos\varphi) d\varphi = [cos\varphi\ sin\varphi]_0^\pi - \int_0^\pi (-sin\varphi)(sin\varphi) d\varphi = 0 - \left(-\int_0^\pi (sin\varphi)^2 d\varphi\right) =$ $\int_0^\pi (sin\varphi)^2 d\varphi = \int_0^\pi 1 - (cos\varphi)^2 d\varphi = \pi - \int_0^\pi (cos\varphi)^2 d\varphi$, also $\int_0^\pi (cos\varphi)^2 d\varphi = \frac{\pi}{2}$.

Also gilt $\int_\Omega x_1^2(x_1^2 + x_2^2)^2 dx = \frac{\pi}{16}$.